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COMPARISON OF REPLACEMENT POLICIES, AND RENEWAL THEORY IMPLICATIONS

by

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1. Introduction and Preliminaries. Among the most useful replacement policies currently in popular use are the age replacement policy and the block replacement policy. Under an age replacement policy a unit is replaced upon failure or at age T, a specified positive constant, whichever comes first. Under a block replacement policy a unit is replaced upon failure and at times $T, 2T, 3T, \ldots$. Block replacement is easier to administer since the planned replacements occur at regular intervals and so are readily scheduled. This type of policy is commonly used with digital computers and other complex electronic systems. On the other hand, age replacement seems more flexible since under this policy planned replacement takes into account the age of the unit. It is therefore of some interest to compare these two policies with respect to the number of failures, number of planned replacements, and number of removals. ("Removal" refers to both failure replacement and planned replacement.)

Block replacement policies have been investigated by E. L. Welker, 1959, R. F. Drenick, 1960, and B. J. Flehinger, 1962. Age replacement policies have been studied by G. Weiss, 1956, and Barlow and Proschan, 1962, among others.

The results of this paper depend heavily on the properties of distributions with monotone failure rate (Barlow, Marshall, and Proschan, 1963). If a unit failure distribution F has a density f, it can be verified by differentiating $\log \bar{F}(x)$ that the failure rate

$$r(x) = f(x)/\bar{F}(x)$$

is increasing (decreasing) if $\log \bar{F}(x)$ is concave when finite (is convex on $[0,\infty)$). We consistently use $\bar{F}$ for $1 - F$. For mathematical convenience and added generality, we use this concavity (convexity) property as the definition of increasing (decreasing) failure rate whether a density exists or not. We shall refer to increasing failure rate by IFR and decreasing failure rate by DFR. It is also easy to show that F is IFR(DFR) if and only if

$$\frac{F(x + \Delta) - F(x)}{\bar{F}(x)}$$

is increasing (decreasing) for all x such that $\Delta > 0$ and $\bar{F}(x) > 0$. This implies F is IFR(DFR) if and only if

$$\frac{\bar{F}(x - \Delta)}{\bar{F}(x)} \tag{1.1}$$

is increasing (decreasing) in x for all $\Delta > 0$. This property will be needed in Theorem 2.1.

The evaluation of the replacement policies considered also depends heavily on the theory of renewal processes (e.g., Smith, 1958, and Cox, 1962). A renewal process is a sequence $\{X_k\}_{k=1}^{\infty}$ of independent random variables with common distribution F. We also assume $F(0^-) = 0$. If the random variables are not identically distributed we call this a generalized renewal process. Let us write $N(t)$ for the largest value of n for which $X_1 + X_2 + \cdots + X_n \leq t$; in other words $N(t)$ is the number of renewals that will have occurred by time t. We will be primarily concerned with the renewal function, $M(t) = E[N(t)]$.

In this paper we show that, assuming an IFR(DFR) unit failure distribution, the number of failures in $[0,t]$ is stochastically larger (smaller) under an age policy than under a block policy. The number of planned replacements and the total number of removals is always stochastically smaller under an age policy than under a block policy.

By considering the number of failures and the number of removals per unit of time as the duration of the replacement operation becomes indefinitely large, we are able to obtain simple useful bounds on the renewal function. In particular we show that the moments, binomial moments, and variance of an IFR(DFR) renewal process are dominated (subordinated) by the corresponding moments and variance of a Poisson process. Inequalities for generalized renewal processes are also obtained.

ACKNOWLEDGMENT. The basic problems concerning replacement were originally proposed by Igor Bazovsky, 1962; he also conjectured (2.5) and (2.6). We are indebted to Albert W. Marshall and Ronald Pyke for help and advice.

2. <u>Contrast Between Age and Block Replacement</u>. Denote the number of renewals in $[0,t]$ when replacement occurs only at failure by $N(t)$ and let $M(t) = E[N(t)]$. Denote the number of failures in $[0,t]$ under a block policy by $N_B^*(t)$ and under an age policy by $N_A^*(t)$, both having replacement interval T. The following theorem provides a stochastic comparison of the number of failures experienced under these policies. We assume $F(0^-) = 0$ throughout.

<u>Theorem 2.1</u> If F is IFR(DFR), then

$$P[N(t) \geq n] \underset{(\leq)}{\geq} P[N_A^*(t) \geq n] \underset{(\leq)}{\geq} P[N_B^*(t) \geq n] \tag{2.1}$$

for $T \geq 0$, $n = 0, 1, 2, \ldots$. Equality is attained for the exponential distribution $F(x) = 1 - e^{-x/\mu_1}$ where μ_1 denotes the mean of F.

We defer the proof of Theorem 2.1 to Section 3. The following useful bounds on $M(t)$ are an immediate consequence of Theorem 2.1.

<u>Corollary 2.1</u> If F is IFR(DFR), then

(i) $M(t) \underset{(\leq)}{\geq} E[N_A^*(t)] \underset{(\leq)}{\geq} E[N_B^*(t)]$

(ii) $M(t) \underset{(\leq)}{\geq} kM(t/k) \qquad k = 1,2,\ldots$ (2.2)

(iii) $M(t) \underset{(\geq)}{\leq} t/\mu_1$

(iv) $M(h) \underset{(\geq)}{\leq} M(t + h) - M(t) \qquad h \geq 0$

for all $t \geq 0$. The inequalities are sharp.

Proof

(i) follows from Theorem 2.1 and the fact that

$$M(t) = E[N(t)] = \sum_{n=1}^{\infty} P[N(t) \geq n].$$

(ii) Let $T = t/k$ and observe that for this replacement interval

$$M(t) \underset{(\leq)}{\geq} E[N_B^*(t)] = kM(T) = kM(t/k).$$

(iii) By (ii)

$$\frac{M(kT)T}{kT} \underset{(\leq)}{\geq} M(T) \qquad k = 1,2,3,\ldots.$$

Letting $k \to \infty$ we obtain

$$M(T) \underset{(\geq)}{\leq} T/\mu_1,$$

since

$$\lim_{t\to\infty} \frac{M(t)}{t} = 1/\mu_1$$

by the fundamental renewal theorem (e.g., Smith, 1958).

(iv) Using the notation of Theorem 2.1

$$M(t + h) - M(t) = \int_0^t \int_0^h [1 + M(h - u)]dF_x(u)d_xP[\delta(t) \leq x]$$

$$\underset{(\leq)}{\geq} \int_0^t \int_0^h [1 + M(h - u)]dF(u)d_xP[\delta(t) \leq x]$$

since $F_x(u)$ is increasing (decreasing) in x. Therefore

$$M(t + h) - M(t) \underset{(\leq)}{\geq} M(h)\int_0^t d_xP[\delta(t) \leq x] = M(h). \quad ||$$

The following formula, true for all distributions with second moment $\mu_2 < \infty$, provides an interesting comparison with (2.2)

$$M(t) = t/\mu_1 + \mu_2/2\mu_1^2 - 1 + o(1).$$

(see e.g. Smith, 1958). As we shall show, inequality (2.2) can be strengthened by assuming somewhat more. It is also true under weaker assumptions.

Let $N_A(t)$ and $N_B(t)$ denote the total number of removals in $[0,t]$ following an age and a block replacement policy respectively. The following theorem, true for all distributions, is intuitively obvious.

Theorem 2.2

$$P[N(t) \geq n] \leq P[N_A(t) \geq n] \leq P[N_B(t) \geq n]$$

for all $t \geq 0$, $n = 0,1,2,\ldots$. We defer the proof to Section 3.

Corollary 2.2

(i) $M(t) \leq E[N_A(t)] \leq E[N_B(t)]$

(ii) $M(t) \leq kM(t/k) + k \qquad k = 1,2,\ldots \qquad (2.3)$

(iii) $M(t) \geq t/\mu_1 - 1$

for all $t \geq 0$.

Proof

(i) is an immediate consequence of Theorem 2.2.

(ii) Let $T = t/k$ and observe that for this replacement interval

$$M(t) \leq E[N_B(t)] = kM(T) + k = kM(t/k) + k.$$

(iii) follows from the elementary renewal theorem

$$\lim_{t\to\infty} M(t)/t = 1/\mu_1. \quad \|$$

The following theorem summarizes some well-known limit results from renewal theory.

Theorem 2.3

(i) $\lim_{t\to\infty} N(t)/t = \lim_{t\to\infty} M(t)/t = 1/\mu_1$

(ii) $\lim_{t\to\infty} N_A^*(t)/t = \lim_{t\to\infty} E[N_A^*(t)]/t = F(T)/\int_0^T \bar{F}(x)dx$

(iii) $\lim_{t\to\infty} N_B^*(t)/t = \lim_{t\to\infty} E[N_B^*(t)]/t = M(T)/T.$

Proof

(i) See e.g. Smith, 1958.

(ii) The times between failures $\{Y_i\}_{i=1}^{\infty}$ for an age replacement policy constitute a renewal process with distribution

$$\bar{H}_T(t) = P[Y_i \geq t] = [\bar{F}(T)]^n \bar{F}(t - nT) \tag{2.4}$$

for $nT \leq t < (n + 1)T$. The expected value of Y_i can be calculated from (2.4) to be

$$E[Y_i] = \int_0^T \bar{F}(x)dx/F(T).$$

(Weiss, 1956, calculated higher moments.) Hence

$$\lim_{t\to\infty} N_A^*(t)/t = F(T)/\int_0^T \bar{F}(x)dx$$

by (i).

(iii) Let $N_{B_i}^*(T)$ denote the number of failures in $[(i - 1)T, iT]$

following a block replacement policy. Then

$$\lim_{t\to\infty} N^*(t)/t = \lim_{n\to\infty} \sum_{i=1}^{n} N^*_{B_i}(T)/nT = E[N^*_{B_i}(T)]/T = M(T)/T. \quad \|$$

For a generalization of (iii), see, e.g., Flehinger, 1962.

Using these limit results we can sharpen (2.2).

Theorem 2.4

(i) $$M(t) \geq t/\int_0^t \bar{F}(x)dx - 1 \geq t/\mu_1 - 1. \tag{2.5}$$

(ii) If F is IFR(DFR), then

$$M(t) \underset{(\geq)}{\leq} tF(t)/\int_0^t \bar{F}(x)dx \underset{(\geq)}{\leq} t/\mu_1 \tag{2.6}$$

for all $t \geq 0$.

Proof

(i) By Corollary 2.2 (i)

$$E[N_A(t)] \leq E[N_B(t)].$$

By Theorem 2.3 (ii) and (iii)

$$\lim_{t\to\infty} E[N_A(t)]/t = 1/\int_0^T \bar{F}(x)dx \leq \lim_{t\to\infty} E[N_B(t)]/t = M(T)/T + 1/T,$$

which implies

$$M(T) \geq T/\int_0^T \bar{F}(x)dx - 1 \qquad \text{for all } T > 0.$$

Obviously $\int_0^T \bar{F}(x)dx \leq \mu_1$ implies $T/\int_0^T \bar{F}(x)dx - 1 \geq T/\mu_1 - 1$.

(ii) By Corollary 2.1 (i)

$$E[N_A^*(t)] \underset{(\le)}{\ge} E[N_B^*(t)].$$

By Theorem 2.3 (ii) and (iii)

$$M(T)/T = \lim_{t\to\infty} E[N_B^*(t)]/t \underset{(\ge)}{\le} \lim_{t\to\infty} E[N_A^*(t)]/t = F(T)/\int_0^T \bar{F}(x)dx \underset{(\ge)}{\le} 1/\mu_1.$$

The last inequality follows by noting that $\bar{H}_T(t)$ [see (2.4)] is decreasing (increasing) in T if F is IFR(DFR). ||

Equality is attained in (2.5) at t^- where $t = k\mu_1$ $(k = 1,2,\ldots)$ by the distribution degenerate at the mean μ_1 and of course $M(t) \ge 0$ is sharp for $t < \mu_1$. Equality is attained in (2.6) for the Poisson process. Note that inequality (2.6) is an improvement on (2.5) in the DFR case since

$$tF(t)/\int_0^t \bar{F}(x)dx \ge \int_0^t F(x)dx/\int_0^t \bar{F}(x)dx = t/\int_0^t \bar{F}(x)dx - 1.$$

From (2.5) and (2.6) we see that

$$1/\mu_1 - 1/T \le M(T)/T \le F(T)/\int_0^T \bar{F}(x)dx \le 1/\mu_1$$

when F is IFR. Hence in this circumstance, the expected number of failures per unit time in the limit does not differ by more than one unit using either policy.

3. Proofs of the Theorems of Section 2.

Proof of Theorem 2.1: Assume F is IFR. First let us suppose $0 \leq t \leq T$, where T is the replacement interval. Let $P[N(t) \geq n|x]$ denote the probability that $N(t) \geq n$, given that the age of the unit in operation at time 0 is x. Then we shall show that

$$P[N(t) \geq n|x] \geq P[N_A^*(t) \geq n|x] \geq P[N_B^*(t) \geq n]. \tag{3.1}$$

For $n = 0$, (3.1) is trivially true. For $n > 0$, we can rewrite (3.1) as

$$\int_0^t F^{(n-1)}(t - u)dF_x(u) \geq \int_0^t F^{(n-1)}(t - u)dF_x^T(u) \geq \int_0^t F^{(n-1)}(t - u)dF(u) \tag{3.1'}$$

where $F^{(n)}(t)$ denotes the n-fold convolution of F with itself and

$$F_x(u) = \frac{F(x + u) - F(x)}{\bar{F}(x)} .$$

$F_x^T(u)$ is the distribution of the time to the first failure when the age of the unit in operation at time 0 is x and planned replacement is scheduled for $T - x$ if no failure intervenes. We need specify the distribution $F_x(u)$ only on $[0,t]$:

$$F_x^T(u) = \begin{cases} \dfrac{F(x + u) - F(x)}{\bar{F}(x)} & \text{if } u \leq T - x \\[2ex] \dfrac{F(T) - F(x) + \bar{F}(T)F(u - T + x)}{\bar{F}(x)} & \text{if } T - x \leq u \leq t. \end{cases}$$

To prove (3.1') we need only show

$$F_x(u) \geq F_x^T(u) \geq F(u) \qquad \text{for} \quad 0 \leq u \leq t \tag{3.2}$$

since $F^{(n-1)}(t - u)$ is decreasing in u. For $u \leq T - x$

$$F_x(u) = F_x^T(u) = \frac{F(x + u) - F(x)}{\overline{F}(x)} \geq F(u)$$

since F is IFR. For $t - x \leq u \leq t$

$$\frac{F(x + u - T + T) - F(T)}{\overline{F}(T)} \geq F(x + u - T)$$

implies

$$F(x + u) \geq F(T) + \overline{F}(T) F(u - T + x)$$

and so

$$F_x(u) = \frac{F(x + u) - F(x)}{\overline{F}(x)} \geq \frac{\overline{F}(x) - \overline{F}(T) + \overline{F}(T) F(u - T + x)}{\overline{F}(x)} = F_x^T(u)$$

proves the first inequality in (3.2). Also for $T - x < u \leq t$,

$$\frac{\overline{F}(u - T + x)}{\overline{F}(u)}$$

is increasing in u by (1.1) since we may assume $x \leq T$. Therefore

$$\frac{\overline{F}(u - T + x)}{\overline{F}(u)} \leq \frac{\overline{F}(x)}{\overline{F}(T)}$$

since $0 \leq u \leq T$. Rearrangement yields

$$\overline{F}(u) \geq \frac{\overline{F}(T)\overline{F}(u - T + x)}{\overline{F}(x)},$$

so that

$$F_x^T(u) = \frac{\bar{F}(x) - \bar{F}(T) + \bar{F}(T)F(u - T + x)}{\bar{F}(x)} \geq F(u)$$

which completes the proof of (3.2). From (3.2) we deduce that for $x > 0$ and $0 \leq t \leq T$

$$P[N(t) \geq n|x] \geq P[N_A^*(t) \geq n|x] \geq P[N_B^*(t) \geq n].$$

Now suppose $kT < t \leq (k + 1)T$, where $k \geq 1$. The proof proceeds by induction on k. Assume (3.1) is true for $0 \leq t \leq kT$. For $n = 0$, (3.1) is trivially true. For $n > 0$, write

$$P[N(t) \geq n] = \sum_{r=0}^{n} \int_0^T \{P[N(T) = r|\delta(T) = x]P[N(t-T) \geq n-r|\delta(T) = x]\}d_x P[\delta(T) \leq x]$$

$$P[N_A^*(t) \geq n] = \sum_{r=0}^{n} \int_0^T \{P[N_A^*(T) = r|\delta(T) = x]P[N_A^*(t-T) \geq n-r|\delta(T) = x]\}d_x P[\delta(T) \leq x]$$

and

$$P[N_B^*(t) \geq n] = \sum_{r=0}^{n} \int_0^T \{P[N_B^*(T) = r|\delta(T) = x]P[N_B^*(t-T) \geq n-r]\}d_x P[\delta(T) \leq x]$$

where $\delta(T)$ is a random variable denoting the age of the unit in use at time T. By inductive hypothesis

$$P[N(t-T) \geq n-r|\delta(T) = x] \geq P[N_A^*(t-T) \geq n-r|\delta(T) = x] \geq P[N_B^*(t-T) \geq n-r].$$

Also

$$P[N(T) = r|\delta(T) = x] = P[N_A^*(T) = r|\delta(T) = x] = P[N_B^*(T) = r|\delta(T) = x]$$

since all three policies coincide on $[0,T]$. Hence (3.1) follows for $kT \leq t \leq (k + 1)T$ for all $k \geq 1$ by the axiom of mathematical induction.

For F DFR the proof is similar with the inequalities reversed. ||

<u>Proof of Theorem 2.2</u> (Due to Albert W. Marshall) Let $\{X_k\}_{k=1}^{\infty}$ denote a realization of the lives of successive components. We shall compute what would have occurred under an age and under a block replacement policy. Let $T_A^n(T_B^n)$ denote the time of the n^{th} removal under an age (block) replacement policy. Then

$$T_A^n = \min(T_A^{n-1} + T, T_A^{n-1} + X_n)$$

$$T_B^n = \min(T_B^{n-1} + \alpha, T_B^{n-1} + X_n)$$

where $\alpha (0 \leq \alpha \leq T)$ is the remaining life to a scheduled replacement. Since initially $T_A^1 = T_B^1$, we have by induction $T_A^n \geq T_B^n$. Thus for any realization $\{X_k\}_{k=1}^{\infty}$ $N_A(t)$ is smaller than $N_B(t)$. By a similar argument $N(t)$ is smaller than $N_A(t)$ for any realization. ||

4. Renewal Theory Consequences. A renewal process is an IFR(DFR) renewal process if the underlying distribution F is IFR(DFR). This does not imply that $N(t)$, the renewal quantity associated with an IFR(DFR) renewal process, is IFR(DFR). (See Barlow, Marshall, Proschan, 1963). However, just as the geometric (exponential) distribution is a natural comparison distribution for IFR and DFR discrete (continuous) random variables, the Poisson process serves as a natural comparison process for IFR and DFR renewal processes. In Corollary 2.1 we saw that the mean of an IFR(DFR) process is dominated (subordinated) by the mean of an associated Poisson process. This is also true of the binomial moments and, indeed, even the variance.

We define the m^{th} binomial moment, $B_m(t)$, as

$$B_m(t) = \sum_{j=0}^{\infty} \binom{j}{m} P[N(t) = j].$$

The following result is no doubt well known. However since we cannot cite a reference we present a short proof.

Lemma 4.1 For any renewal process $\{N(t);\ t \geq 0\}$,

$$B_m(t) = M^{(m)}(t)$$

where $M^{(m)}(t)$ denotes the m-fold convolution of $M(t) = E[N(t)]$.

Proof Let $B_m^*(s) = \int_0^\infty e^{-st} dB_m(t)$. Then

$$B_m^*(s) = \sum_{j=0}^{\infty} \binom{j}{m} \{[F^*(s)]^j - [F^*(s)]^{j+1}\}.$$

Now

$$\sum_{j=0}^{\infty} \binom{j}{m} x^j = \frac{x^m}{(1-x)^{m+1}} \qquad \text{for } |x| < 1$$

implies

$$B_m^*(s) = \frac{[F^*(s)]^m}{[1 - F^*(s)]^m} = [M^*(s)]^m.$$

The result follows by the inversion theorem for Laplace transforms. ||

The mean life of a used unit of age t

$$\int_t^{\infty} \bar{F}(x)\,dx/\bar{F}(t)$$

is called the mean residual life of the unit. If F is IFR(DFR) with mean μ_1, then

$$\int_t^{\infty} \bar{F}(x)\,dx/\bar{F}(t) \underset{(\geq)}{\leq} \mu_1. \tag{4.1}$$

Of course, the converse is not true and (4.1) is a significant weakening of the IFR(DFR) assumption. The following proof is due to R. Pyke.

<u>Theorem 4.1</u> If F has mean μ_1 and for $0 \leq t < \infty$

$$\int_t^{\infty} \bar{F}(x)\,dx/\bar{F}(t) \underset{(\geq)}{\leq} \mu_1,$$

then

$$B_m(t) \underset{(\geq)}{\leq} \frac{t^m}{m!(\mu_1)^m} \qquad \text{for } m = 0,1,2,\ldots, 0 \leq t < \infty,$$

and $M_m(t) \underset{(\geq)}{\leq} M_m^P(t)$, where $M_m(t)$ is the m^{th} moment of the IFR renewal process and $M_m^P(t)$ is the m^{th} moment of the corresponding Poisson process.

<u>Proof</u> Let $\{X_k\}_{k=1}^{\infty}$ be a renewal process with corresponding distribution F. Let

$$\gamma(t) = X_1 + X_2 + \cdots + X_{N(t)+1} - t$$

and

$$\delta(t) = t - [X_1 + X_2 + \cdots + X_{N(t)}].$$

Then

$$P[\gamma(t) \geq u] = \int_0^t \overline{F}_x(u) d_x P[\delta(t) \leq x]$$

where

$$F_x(u) = \frac{F(x+u) - F(x)}{\overline{F}(x)} .$$

Therefore

$$\begin{aligned} E[\gamma(t)] &= \int_0^\infty P[\gamma(t) \geq u] du \\ &= \int_0^\infty \left[\int_{0^-}^t P[\gamma(t) \geq u \mid \delta(t) = x] d_x P[\delta(t) \leq x] \right] du \\ &= \int_0^\infty \left[\int_{0^-}^t \overline{F}_x(u) d_x P[\delta(t) \leq x] \right] du \\ &= \int_{0^-}^t \left[\int_0^\infty \overline{F}_x(u) du \right] d_x P[\delta(t) \leq x] \end{aligned}$$

by Fubini's theorem. Since by hypothesis

$$\int_x^\infty \overline{F}(u) du / \overline{F}(x) \underset{(\geq)}{\leq} \mu_1$$

we have

$$E[\gamma(t)] \underset{(\geq)}{\leq} \int_{0^-}^t \mu_1 d_x P[\delta(t) \leq x] = \mu_1 .$$

But

$$E[\gamma(t)] = \mu_1 [M(t) + 1] - t$$

implies

$$M(t) \underset{(\geq)}{\leq} t/\mu_1 .$$

The first result follows from Lemma 4.1 and convolution.

To obtain the second result note that $x^n = \sum_{m=1}^{n+1} x(x-1)\cdots(x-m+1)S_n^m$, where S_n^m are the Stirling numbers of the second kind, Jordan, 1950, p. 168. As pointed out by Jordan, 1950, p. 169, the S_n^m are positive. The second result thus follows. ||

From Lemma 4.1 we can compute the variance of the renewal quantity, namely

$$\mathrm{Var}[N(t)] = 2\int_0^t M(t-x)dM(x) + M(t) - [M(t)]^2.$$

Using this formula we can prove

<u>Theorem 4.2</u> If $\{N(t);\ t \geq 0\}$ is an IFR(DFR) renewal process, then

$$\mathrm{Var}[N(t)] \underset{(\geq)}{\leq} E[N(t)] = M(t).$$

The inequality is sharp.

<u>Proof</u> Assume $\{N(t);\ t \geq 0\}$ is an IFR renewal process. Since

$$\mathrm{Var}[N(t)] = \int_0^t [2M(t-x) + 1 - M(t)]dM(x)$$

we need only show

$$\int_0^t [2M(t-x) - M(t)]dM(x) \leq 0.$$

But $M(x) \leq M(t) - M(t-x)$ by (iv) of Corollary 2.1 implies that we need only show

$$\int_0^t [M(t-x) - M(x)]dM(x) \leq 0.$$

Clearly

$$\int_0^t [M(t-x) - M(x)]dM(x) = \int_0^{t/2} [M(t-x) - M(x)]dM(x) + \int_{t/2}^t [M(t-x) - M(x)]dM(x).$$

Let $y = t - x$, then

$$\int_{t/2}^{t} [M(t - x) - M(x)]dM(x) = \int_{0}^{t/2} [M(t - y) - M(y)]dM(t - y).$$

Hence we need only show

$$\int_{0}^{t/2} [M(t - x) - M(x)]dM(x) \leq \int_{0}^{t/2} [M(t - x) - M(x)]d[M(t) - M(t - x)].$$

This follows immediately, since $M(t - x) - M(x)$ is non-increasing in x, $[M(t - x) - M(x)] \geq 0$ for $0 \leq x \leq t/2$ and

$$M(x) \leq M(t) - M(t - x).$$

All inequalities are reversed if F is DFR. Equality is attained by the Poisson process. ||

Next we obtain a generalization of the inequality

$$M(t) \underset{(\geq)}{\leq} t/\mu_1$$

which holds when successive replacements have <u>different</u> failure distributions but a common mean. The method of proof is quite different from that used in Theorem 2.1 or Theorem 4.1. We will need to define the generalized renewal function

$$M_o(t) = F_1(t) + F_1{*}F_2(t) + F_1{*}F_2{*}F_3(t) + \ldots . \tag{4.2}$$

Note that $M_o(t)$ is the expected number of renewals in a stochastic process in which the first unit has distribution F_1, its replacement has distribution F_2, etc.

Theorem 4.3 Let $F_1, F_2, F_3, \dots$ be non-degenerate IFR(DFR) distributions with common mean μ_1 and assume

$$F_i(t) \neq G(t) = 1 - e^{-t/\mu_1} \qquad \text{for} \quad t \geq 0$$

and $i = 1,2,\dots$. Then

$$M_o(t) \underset{(>)}{<} t/\mu_1 \qquad \text{for} \quad t > 0. \quad (4.3)$$

Proof Assume F_i $(i = 1,2,\dots)$ are IFR. First suppose $F_1 = F_2 = \dots$. Then

$$M_o(t) = \sum_{k=1}^{\infty} F_1^{(k)}(t) < \sum_{k=1}^{\infty} G^{(k)}(t) = t/\mu_1$$

for $0 < t \leq \mu_1$. (See Barlow, Marshall, 1963). Suppose there exists $\tau > \mu_1$ such that $M_o(\tau) = \tau/\mu_1$. Then, since F has mass in $[0,\tau]$

$$\tau/\mu = M_o(\tau) = \int_0^{\tau} [1 + M_o(\tau - x)]dF_1(x) < \int_0^{\tau} [1 + \frac{\tau - x}{\mu_1}]dF_1(x)$$

or

$$\tau/\mu_1 < F_1(\tau) + \int_0^{\tau} F_1(x)dx/\mu_1$$

which implies

$$[\int_0^{\tau} \overline{F}_1(x)dx/F_1(\tau)] < \mu_1.$$

But this contradicts (2.6) of Theorem 2.4. Hence $M_o(\tau) < \tau/\mu_1$, $0 < \tau < \infty$, for this special case.

The argument proceeds by induction. Suppose the theorem is true for all sequences of distributions of the form $H_1, H_2, \dots, H_k = H_{k+1} = H_{k+2} = \dots$ where the H_i satisfy the IFR assumption.

$$M_1(t) = F_2(t) + F_2*F_3(t) + \cdots + F_2*F_3*\cdots*F_{k+1}(t)$$

$$+ F_2*F_3*\cdots*F_{k+1}*F_{k+1}(t) + \cdots$$

and

$$M_o(t) = \int_0^t [1 + M_1(t - x)]dF_1(x).$$

As before, $M_o(t) < t/\mu_1$ for $0 < t \leq \mu_1$. Suppose there exists $\tau > \mu_1$ such that $M_o(\tau) = \tau/\mu_1$. This implies

$$\tau/\mu_1 = \int_0^\tau [1 + M_1(\tau - x)]dF_1(x) < F_1(\tau) + \int_0^\tau F_1(x)dx$$

and

$$\int_0^\tau \overline{F}_1(x)/F_1(\tau) < \mu_1.$$

This is a contradiction. Theorem 4.3 follows by the axiom of mathematical induction.

All inequalities are reversed for DFR distributions. ||

The method of proof used in Theorem 4.3 can be used to generalize the bound on $M(t)$ in yet another direction.

Theorem 4.4 Assume F has density f, failure rate $r(x) = f(x)/[\overline{F}(x)]$, and mean μ_1.

(i) If $r(x) \geq \alpha$ for all x, then $M(t) \leq t/\mu_1 + 1/\alpha\mu_1 - 1$.

(ii) Suppose there exists $\delta > 0$ such that $f(x) > 0$ for $0 < x < \delta$. If $r(x) \leq \beta$, then $M(t) \geq t/\mu_1 + 1/\beta\mu_1 - 1$.

All inequalities are sharp.

<u>Proof</u> If $F(x) = 1 - e^{-x/\mu_1}$ the bounds are attained. Hence suppose $F(x) \not\equiv 1 - e^{-x/\mu_1}$. Then

$$\inf_x r(x) < 1/\mu_1 < \sup_x r(x) \tag{4.4}$$

(see Barlow, Marshall, Proschan, 1963). (4.4) implies $\alpha < 1/\mu_1 < \beta$, and

$$1/\alpha\mu_1 - 1 > 0$$
$$1/\beta\mu_1 - 1 < 0.$$

Since $M(0) = 0$,

$$t/\mu_1 + 1/\mu_1\beta - 1 < M(t) < t/\mu_1 + 1/\alpha\mu_1 - 1$$

for t sufficiently small.

(i) If $\alpha = 0$, we are done. Hence assume $\alpha > 0$ and suppose there exists $0 < \tau < \infty$ such that

$$M(\tau) = \tau/\mu_1 + 1/\alpha\mu_1 - 1$$

and

$$M(t) < t/\mu_1 + 1/\alpha\mu_1 - 1 \qquad \text{for } t < \tau.$$

Then

$$\tau/\mu_1 + 1/2\mu_1 - 1 = M(\tau) = \int_0^\tau [1 + M(\tau - x)]dF(x) < \int_0^\tau [(\tau - x)/\mu_1 + 1/\alpha\mu_1]dF(x) \tag{4.5}$$

since $r(x) \geq \alpha$ implies $F(t) \geq 1 - e^{-\alpha t}$ and hence F has mass in $[0,\tau]$.
(4.5) implies

$$\tau/\mu_1 + 1/\alpha\mu_1 - 1 < \frac{1}{\alpha\mu_1}F(\tau) + \int_0^\tau \frac{F(x)}{\mu_1}dx.$$

But $r(x) \geq a$ implies

$$f(x) \geq a\overline{F}(x)$$

and so

$$\overline{F}(\tau) = \int_\tau^\infty f(x)dx \geq a\int_\tau^\infty \overline{F}(x)dx$$

which implies

$$\frac{1}{a\mu_1}F(\tau) + \int_0^\tau \frac{F(x)dx}{\mu_1} \leq \frac{\tau}{\mu_1} + \frac{1}{a\mu_1} - 1,$$

a contradiction. Hence, actually

$$M(t) < \frac{t}{\mu_1} + \frac{1}{a\mu_1} - 1$$

when $F(t) \not\equiv 1 - e^{-t/\mu_1}$.

(ii) If $\beta = \infty$, the inequality follows from Corollary 2.2 (iii). Hence suppose $\beta < \infty$. There exists $0 < \tau < \infty$ such that

$$M(\tau) = \frac{\tau}{\mu_1} + \frac{1}{\beta\mu_1} - 1$$

and

$$M(t) > \frac{t}{\mu_1} + \frac{1}{\beta\mu_1} - 1 \qquad \text{for } t < \tau.$$

Then

$$\frac{\tau}{\mu_1} + \frac{1}{\beta\mu_1} - 1 = M(\tau) = \int_0^\tau [1 + M(\tau - x)]dF(x) > \int_0^\tau [\frac{\tau - x}{\mu_1} + \frac{1}{\beta\mu_1}]dF(x)$$

since F has mass in $[0,\tau]$. Therefore

$$\frac{\tau}{\mu_1} + \frac{1}{\beta\mu_1} - 1 > \frac{F(\tau)}{\beta\mu_1} + \int_0^\tau \frac{F(x)dx}{\mu_1} \geq \frac{\tau}{\mu_1} + \frac{1}{\beta\mu_1} - 1$$

since $r(x) \leq \beta$. This is a contradiction and therefore

$$M(t) > \frac{t}{\mu_1} + \frac{1}{\beta\mu_1} - 1$$

for $0 < t < \infty$ when $F(t) \not\equiv 1 - e^{-t/\mu_1}$. ||

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